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A test for the integrability of nonlinear evolution equations

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Abstract. Most of the known integrable nonlinear evolution equations have recursion operators which generate an infinity of infinitesimal transformations about any solution. Using this property we derive a simple criterion for the integrability of such an equation. The recursion operator also generates a hierarchy of related nonlinear evolution equations.

1. Introduction

A large class of nonlinear evolution equations (NLEE) solvable by the inverse spectral transform have been derived starting from a system of coupled linear equations (Ablowitz *et al* 1974, Newell 1979, Calogero and Degasperis 1976, 1977, Konopelchenko 1980). The converse problem of deciding the integrability of a given NLEE is of great interest. One such method (Wahlquist and Estabrook 1975) exploits the fact that most of the known integrable NLEE arise as integrability conditions of a system of Pfaffians. See Kaup (1980) for a tutorial presentation of the method. The main difficulty in this method is the closure of the Lie algebra that is naturally generated in their approach. This difficulty has been overcome by Corones and Testa (1976) who have argued that to obtain the Bäcklund transformation and the associated spectral problem of a NLEE it is sufficient to consider just one pseudopotential. Another approach, due to Chen *et al* (1979), exploits the existence of an infinity of conservation laws for an integrable NLEE. They prove the existence of four or five conserved quantities for a given NLEE, and infer the existence of an infinity of conserved quantities and hence the integrability of the NLEE.

We prove the integrability of the NLEE

$$u_t + K(u) = 0 \tag{1.1}$$

by proving the existence of an infinity of infinitesimal transformations (IT) about any solution $u(x, t)$ of (1.1). This is established by proving the existence of a recursion operator $T(u)$, starting from (1.1), and noting that $u_x(x, t)$ is always an IT about $u(x, t)$. Then $T(u)$ acting repeatedly on $u_x(x, t)$ gives an infinity of IT (hence the nomenclature for T). The form of $T(u)$ (with some undetermined constants) is found by noting that both $u_x(x, t)$ and $K(u)$ are IT about $u(x, t)$. The constants in $T(u)$ are determined from the condition that if $y(x, t)$ is any IT so is $T(u)\{y(x, t)\}$. This condition leads to equation (2.8), which is the equation obeyed by the Lax pairs (Lax 1968), though the operators are different.

The importance of recursion operators has been discussed by various authors (Olver 1977, Wadati 1978, Fuchssteiner 1979, Fokas 1980, Ibragimov and Shabat 1980, Fuchssteiner and Fokas 1981). It is clear from these that recursion operators also give another viewpoint to the study of integrable NLEE. Equation (2.8) has been derived by Fuchssteiner (1979) and Fokas (1980) though they have not used it to find the recursion operator. To our knowledge only Fokas (1980), starting from a NLEE of the form (1.1), has derived its recursion operator. He has used the properties of the generators of the Lie–Bäcklund transformation to obtain conditions on classes of NLEE so that they have recursion operators. Our work gives another method for finding the recursion operator.

To the question whether a recursion operator, when it exists, will always connect u_x and $K(u)$, we feel that it will be true only for NLEE arising from 2×2 scattering equations. In support of this conclusion we have the following two cases.

The fifth-order NLEE (Sawada and Kotera 1974)

$$u_t + 30u_xu_{xx} + 30uu_{3x} + 180u^2u_x + u_{5x} = 0 \tag{1.2}$$

has a third-order scattering equation (Dodd and Gibbon 1977, Satsuma and Kaup 1977) and the recursion operator is of the sixth order (Fuchssteiner and Oevel 1982). T acting on u_x therefore does not give $K(u)$, but acting on u_x and $K(u)$ gives rise to two sets of IT. We have recently obtained a similar result (details to be published) for the coupled KdV equation (Hirota and Satsuma 1981)

$$\begin{aligned} u_t &= \tfrac{1}{2}u_{3x} + 3uu_x - 6\phi\phi_x \equiv K_1(u, \phi), \\ \phi_t &= -\phi_{3x} - 3u\phi_x \equiv K_2(u, \phi). \end{aligned} \tag{1.3}$$

There is no recursion operator connecting the IT (u_x, ϕ_x) to the IT $(K_1(u, \phi), K_2(u, \phi))$. There is a fourth-order recursion operator T (a 2×2 matrix) connecting (u_x, ϕ_x) to the next fifth-order IT. It is also known that the linear scattering equation associated with (1.3) is of the fourth order (Dodd and Fordy 1982). More work on NLEE arising from higher-order scattering equations is necessary.

In § 2 we briefly develop the formalism and compare equation (2.8) with the Lax equation. The method of § 2 is applied to a NLEE in § 3 and a related hierarchy of NLEE generated.

2. Development of the formalism

If $y(x, t)$ is an IT about a solution $u(x, t)$ of (1.1), i.e. $u(x, t) + \varepsilon y(x, t)$ is also a solution of (1.1) to terms linear in ε, then

$$u_t + \varepsilon y_t + K(u + \varepsilon y, u_x + \varepsilon y_x, \ldots) = 0. \tag{2.1}$$

The evolution equation for $y(x, t)$ is then

$$y_t + y\,\partial K/\partial u + y_x\,\partial K/\partial u_x + \ldots = 0. \tag{2.2}$$

Equation (2.2) can be cast in the form

$$[\partial/\partial t + \hat{A}(u)]\{y\} = 0. \tag{2.3}$$

This defines the operator $\hat{A}$, a linear operator depending on $u(x, t)$ and its spatial derivatives.

To obtain the recursion operator $T(u)$ for the IT we note that $u_x(x, t)$ and $K(u)$ are solutions of (2.2), i.e. are IT about $u(x, t)$. We now look for the most general linear integro-differential operator $T(u)$ such that

$$T(u)\{u_x\} = K(u). \tag{2.4}$$

This operator will have arbitrary constants and these are determined by the requirement that if $y(x, t)$ be any function satisfying (2.3) then so is $T(u)\{y\}$. Thus we require that

$$y_t + \hat{A}(u)\{y\} = 0 \tag{2.5}$$

implies

$$(\partial/\partial t)[T(u)\{y\}] + \hat{A}(u)\{T(u)\{y\}\} = 0. \tag{2.6}$$

From (2.5) and (2.6) we obtain

$$T_t\{y\} = (T\hat{A} - \hat{A}T)\{y\}. \tag{2.7}$$

A sufficient condition for (2.7) is

$$T_t = [T, \hat{A}]. \tag{2.8}$$

If the IT $y(x, t)$ form a complete set, as is the case (Aiyer 1982) for IT about n-soliton solutions of the KdV, then (2.8) is also necessary for (2.7) to be true.

Equation (2.8) is the main equation for the integrability of (1.1). The operator $\hat{A}$ can always be obtained for any NLEE, whereas T can be obtained in many cases.

We now establish the relation between the Lax pair (L, A) and the pair $(T, \hat{A})$. The pair (L, A) satisfies

$$L_t = [L, A], \tag{2.9}$$

where

$$L\psi = \zeta\psi \tag{2.10}$$

is the eigenvalue equation of the linear spectral problems and

$$\psi_t = A\psi \tag{2.11}$$

describes the time variation of ψ. On the other hand the adjoint $T^+(u)$ of the operator $T(u)$ has eigenfunctions which are related to the squares of ψ of (2.10). This has been shown for certain NLEE by Ablowitz *et al* (1974) and Newell and Flaschka (1975). The operator $\hat{A}$ describes the time variation of the IT and it has been shown (Aiyer 1982) that the IT about the n-soliton solution of the KdV are related to the spatial derivatives of the squares of ψ. Thus $(T, \hat{A})$ are operators similar to the pair (L, A) but related to the squares of the eigenfunctions of the associated spectral problem. This formalism therefore further stresses the role of the squares of eigenfunctions (Newell 1980, Kaup 1976).

Chen *et al* (1979) have also obtained the pair $(T, \hat{A})$ for a NLEE which we will consider in the next section. They have shown that for this equation the pair satisfies (2.8), and have therefore identified $(T, \hat{A})$ with the Lax pair (L, A). This, from what precedes, is seen not to be correct.

3. Application—a particular example

3.1.

Consider the nonlinear equation

$$u_t - \mathrm{i}u_{xx} + 2|u|^2 u_x = 0. \tag{3.1}$$

The equation for $y(x,t)$, an IT about $u(x,t)$, is

$$y_t - \mathrm{i}y_{xx} + 2(|u|^2 y + uu_x y^* + u^* u_x y) = 0 \tag{3.2}$$

where * denotes complex conjugate. The operator $\hat{A}$ defined in (2.3) is

$$\hat{A}\{y\} \equiv -\mathrm{i}\partial^2 y/\partial x^2 + 2(|u|^2 y_x + uu_x y^* + u^* u_x y). \tag{3.3}$$

To obtain the recursion operator $T(u)$ we note that $u_x(x,t)$ and $-\mathrm{i}u_{xx} + 2|u|^2 u_x$ are IT about any solution $u(x,t)$ of (3.1). The most general linear operator $T(u)$ such that

$$T(u)\{u_x\} = -\mathrm{i}u_{xx} + 2|u|^2 u_x \tag{3.4}$$

is given by

$$y^{(n+1)}(x,t) = T(u)\{y^{(n)}(x,t)\}$$

$$\equiv -\mathrm{i}\partial y^{(n)}/\partial x + \alpha|u|^2 y^{(n)} + \beta u_x \int^x (u^* y^{(n)} + uy^{(n)*})\,\mathrm{d}x_1$$

$$+\rho u \int^x (u_x y^{(n)*} - u_x^* y^{(n)})\,\mathrm{d}x_1 + \gamma u^* u_x \int^x y^{(n)}\,\mathrm{d}x_1 + \delta uu_x \int^x y^{(n)*}\,\mathrm{d}x_1. \tag{3.5}$$

With $y^{(n)} = u_x$ one gets (3.4). With this choice of $y^{(n)}$ the fourth term on the RHS of (3.5) vanishes identically, but does not for a general IT $y^{(n)}(x,t)$. This term therefore has to be included. $\alpha, \beta, \rho, \gamma, \delta$ are constants, possibly complex.

To determine these constants we demand that $y^{(n+1)}(x,t)$ satisfies (3.2) if $y^{(n)}(x,t)$ does, i.e. if $y^{(n)}(x,t)$ is an IT about $u(x,t)$ then so is $y^{(n+1)}(x,t)$. We do not use equation (2.8) as this will require a 2×2 matrix formulation with fields $u(x,t)$ and $u^*(x,t)$.

The rest consists of long but straightforward algebra. We substitute $y^{(n+1)}(x,t)$ given by (3.5) in (3.2). To eliminate $y_t^{(n)}(x,t)$ and $u_t(x,t)$ we use (3.1) and the fact that $y^{(n)}(x,t)$ satisfies (3.2). We equate to zero, separately, the coefficients of $\int y^{(n)}\,\mathrm{d}x$, $\int y^{(n)*}\,\mathrm{d}x$, $y^{(n)}$, $y^{(n)*}$ and their spatial derivatives. Integrals involving spatial derivatives of $y^{(n)}$ are reduced by integrating by parts. A unique and consistent set of solutions for $\alpha, \beta, \rho, \gamma, \delta$ is

$$\alpha = \beta = \rho = 1, \qquad \gamma = \delta = 0. \tag{3.6}$$

A recursion operator $T(u)$ generating the IT about solutions of (3.1) therefore exists and (3.1) is integrable. From (3.5) and (3.6) the recursion operator $T(u)$ acting on $y(x,t)$, an IT about $u(x,t)$, is

$$T(u)\{y(x,t)\}$$

$$= -\mathrm{i}\partial y/\partial x + |u|^2 y + u_x \int^x (u^* y + uy^*)\,\mathrm{d}x_1$$

$$+u \int^x (u_x y^* - u_x^* y)\,\mathrm{d}x_1. \tag{3.7}$$

3.2.

In analogy with the KdV, sine-Gordon and modified KdV equations we can expect

$$u_t+[T(u)]^n\{u_x\}=0 \tag{3.8}$$

to be integrable for $n=0, 1, \ldots$. $[T(u)]^n$ is the nth power of the operator $T(u)$. $n=1$ gives (3.1). $n=2$ gives the NLEE

$$u_t+3|u|^4u_x-3iu^*(u_x)^2-3i|u|^2u_{xx}-u_{xxx}=0. \tag{3.9}$$

We have explicitly verified that the operators $T(u)$ defined by (3.7) generate an infinity of IT about any solution of (3.9).

4. Conclusions

We have shown that starting from a NLEE (1.1), the recursion operator T for the IT can be found by using the facts that u_x and $K(u)$ are IT about any solution $u(x, t)$ of (1.1), and that if $y(x, t)$ is an IT so is $T\{y(x, t)\}$. This provides a simple method for finding the recursion operator for a given NLEE and also proving its integrability. However, the existence of a recursion operator T connecting u_x to $K(u)$ seems to exist only for those integrable NLEE which arise from 2×2 scattering equations.

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